Available online at www.sciencedirect.com
SCIENCE DIRECT®
Physics Letters A 334 (2005) 337–344
PHYSICS LETTERS A
www.elsevier.com/locate/pla

Practical measurement of joint weak values and their connection to the annihilation operator

J.S. Lundeen [a,*], K.J. Resch [b]

[a] *Department of Physics, University of Toronto, 60 St. George Street, Toronto, ON M5S 1A7, Canada*
[b] *Institut für Experimentalphysik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria*

Received 16 August 2004; received in revised form 19 October 2004; accepted 22 November 2004

Available online 10 December 2004

Communicated by P.R. Holland

Abstract

Weak measurements are a new tool for characterizing post-selected quantum systems during their evolution. Weak measurement was originally formulated in terms of von Neumann interactions which are practically available for only the simplest single-particle observables. In the present work, we extend and greatly simplify a recent, experimentally feasible, reformulation of weak measurement for multiparticle observables [Phys. Rev. Lett. 92 (2004) 130402]. We also show that the resulting "joint weak values" take on a particularly elegant form when expressed in terms of annihilation operators.

PACS: 03.65.Ta; 02.50.-r; 06.30.-k; 03.67.-a

Keywords: Weak measurement; Entanglement; Post-selection; Annihilation operator

Weak measurement was originally proposed by Aharonov, Albert and Vaidman (AAV) [1] as an extension to the standard von Neumann ("strong") model of measurement. A weak measurement can be performed by sufficiently reducing the coupling between the measuring device and the measured system. In this case, the pointer of the measuring device begins in a state with enough position uncertainty that any shift induced by the weak coupling is insufficient to distinguish between the eigenvalues of the observable in a single trial. While at first glance it may seem strange to desire a measurement technique that gives less information than the standard one, recall that the entanglement generated between the quantum system and measurement pointer is responsible for collapse of the wavefunction. Furthermore, if multiple trials are performed on an identically-prepared ensemble of systems, one can measure the average shift of the pointer to any precision—this average shift is called the weak value. A surprising characteristic of weak values is

* Corresponding author.
E-mail address: lundeen@physics.utoronto.ca (J.S. Lundeen).

0375-9601/$ – see front matter
doi:10.1016/j.physleta.2004.11.037

that they need not lie within the eigenvalue spectrum of the observable and can even be complex [2–4]. On the other hand, an advantage of weak measurements is that they do not disturb the measured system nor any other simultaneous weak measurements or subsequent strong measurements, even in the case of noncommuting observables. This makes weak measurements ideal for examining the properties and evolution of systems before post-selection and might enable the study of new types of observables. Weak measurements have been used to simplify the calculation of optical networks in the presence of polarization-mode dispersion [5], applied to slow- and fast-light effects in birefringent photonic crystals [6], and bring a new, unifying perspective to the tunneling-time controversy [7, 8]. Hardy's paradox, introduced in Ref. [9], was analyzed in terms of weak values in [10]. In Ref. [11], weak values were used to physically explain the results of the cavity QED experiment described in [12]. The opposing views expressed in Refs. [13,14] on the role of which-path information and the Heisenberg uncertainty principle in the double-slit experiment are reconciled with the use of weak values in [15]. Weak measurement can be considered the best estimate of an observable in a pre- and post-selected system [16].

The von Neumann interaction was originally used to model standard quantum measurement by mathematically describing the coupling between the measured system and the measurement pointer [17]. The interaction couples an observable $\hat{A}$ of the quantum system to the momentum $\hat{P}$ of the pointer,

$$\mathcal{H} = g\hat{A}\hat{P}, \tag{1}$$

where g is the coupling constant which is assumed to be real to keep $\mathcal{H}$ Hermitian. Since $\hat{A}$ and $\hat{P}$ act in different Hilbert spaces we can safely assume they commute. This interaction would be difficult to implement were it not for the fact that typically the measured system itself is used as part of the measurement device. When measuring $\hat{A}$ of a particle an independent degree of freedom of the particle can be used as the pointer. For example, a birefringent crystal can be oriented so that it will displace the position of photon by an amount that depends on the photon's polarization [18]. Here, $\hat{A}$ is the polarization observable and the pointer is the position of the photon. Another example is the Stern–Gerlach apparatus, where $\hat{A}$ is the spin of the particle and the pointer is the momentum of the particle. If such a measurement strategy were not available, one would require a strong controllable interaction between the quantum system and a separate pointer system. This is typically far too technically difficult to implement.

In modern quantum mechanics, we are increasingly interested in a different class of observables than in the above example, in which only a single particle is involved. Often, one would like to measure correlations between observables of distinct particles, like $\hat{S}_1\hat{S}_2$, the spin of particle one times the spin of particle two. Moreover, any experiment that utilizes or directly measures properties of entanglement is based on such observables and so, much of quantum information and quantum optics deal with these composite or joint observables. The exciting results and complex, rich range of features discovered by studies of entanglement suggests that weak measurement of joint observables should also produce valuable and interesting results. In fact, there already exist a few theoretical ideas for weak measurements that center around joint observables, such as Hardy's paradox [10], nonlocality of a single particle [8], and extensions of the quantum box problem [19,20]. We call the weak value of a joint observable the "joint weak value". If the composite observable is a product of N single-particle observables then the weak value is called the "Nth-order joint weak value".

Joint observables are extremely difficult to measure directly with either strong or weak types of measurement. The difficulty lies in the fact that the necessary von Neumann interaction couples two separate observables, and hence particles, to a single pointer. One, therefore, can no longer use the extra degree of freedom on one of the particles as the pointer and so, one requires multiparticle interactions. An approach using multiparticle interactions was outlined in a proposal for a weak measurement experiment with ions but so far there have been no experimental weak measurements of joint observables [21]. On the other hand, experimental strong measurements of joint observables are feasible and even commonplace. This is made possible by employing a different measurement strategy. Instead of measuring the joint observable directly, each single-particle observable is measured simultaneously but separately. For example, instead of measuring $\hat{S}_1\hat{S}_2$ directly we can measure $\hat{S}_1$ and $\hat{S}_2$ separately and then multiply the results trial by trial.

If one wants to strongly measure the joint observable $\hat{A}_1\hat{A}_2\cdots\hat{A}_N = \hat{M}$, instead of using the multiparticle von Neumann Hamiltonian $\mathcal{H} = g\hat{M}\hat{P}$, the general strategy is to simultaneously apply N standard single-particle von Neumann interaction Hamiltonians,

$$\mathcal{H} = g_1\hat{A}_1\hat{P}_1 + g_2\hat{A}_2\hat{P}_2 + \cdots = \sum_{j=1}^{N} g_j\hat{A}_j\hat{P}_j. \tag{2}$$

Given that we can already perform each of the single-particle Hamiltonians, it is straightforward to implement the total Hamiltonian. This strategy allows one to make projective measurements of $\hat{M}$ which is all that is required to measure the expectation value of $\hat{M}$,

$$\langle\hat{M}\rangle = \langle\hat{A}_1\hat{A}_2\cdots\hat{A}_N\rangle \propto \langle\hat{X}_1\hat{X}_2\cdots\hat{X}_N\rangle, \tag{3}$$

where $\hat{X}_i$ is the position operator of the pointer and provided all $\hat{A}_i$ commute. In other words, the expectation value of $\hat{M}$ is related to the correlation between the positions of all N pointers.

In two earlier works, an analogous strategy was applied to weak measurements [22,23]. The Hamiltonian in Eq. (2) is utilized in the weak regime to create correlations in the deflections of the N pointers proportional to the weak value. Specifically, the Nth-order joint weak value was related to two correlations between all N pointer deflections and a complicated combination of lower-order joint weak values. In this work, we show that the Nth-order joint weak value takes on an elegant and simple form closely related to the strong measurement formula in Eq. (3) when expressed entirely in terms of N-particle correlations. This new and simplified form lends itself to a new way of thinking about single and joint weak measurements in terms of expectation values of products of annihilation operators.

We begin by deriving AAVs formula for the weak value of a single-particle observable. AAV based weak measurement on the weak limit of the standard approach to measurement. Specifically, they use the von Neumann interaction in Eq. (1), which we assume to be constant over some interaction time t. The measurement pointer is initially in a Gaussian wavefunction centered at zero,

$$\langle x|\phi\rangle = \phi(x) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^{1/2}\exp\left(-\frac{x^2}{4\sigma^2}\right), \tag{4}$$

where σ is the rms width of $|\phi(x)|^2$. In most experiments, quantum mechanical systems are initially prepared in a known initial state $|I\rangle$. Since this preparation usually involves measuring an ensemble of systems and selecting the subensemble with the correct outcome, this is called pre-selection. For a strong measurement, the von Neumann interaction with a pre-selected system state shifts the mean position of the pointer $\langle\hat{X}\rangle$ by $gt\langle I|\hat{A}|I\rangle$ and leaves $\langle\hat{P}\rangle$ unchanged. AAV considered the case where we further restrict ourselves to the subensemble of system states that are found to be in $|F\rangle$ after the measurement, a procedure called post-selection. A weak measurement performed between the pre- and post-selection can result in very different expectation values than in strong measurements, as we will see.

After the pointer weakly interacts with the initial system-pointer state $|\psi(0)\rangle = |I\rangle|\phi\rangle$ the state evolves to

$$\begin{aligned}\big|\psi(t)\big\rangle &= \exp\left(\frac{-i\mathcal{H}t}{\hbar}\right)|I\rangle|\phi\rangle \\ &= \left(1 - \frac{i\mathcal{H}t}{\hbar} - \cdots\right)|I\rangle|\phi\rangle \\ &= |I\rangle|\phi\rangle - \frac{igt}{\hbar}\hat{A}|I\rangle\hat{P}|\phi\rangle - \cdots.\end{aligned} \tag{5}$$

We project out the part of the state that is post-selected in state $|F\rangle$,

$$\begin{aligned}&\langle F|\exp\left(\frac{-i\mathcal{H}t}{\hbar}\right)|I\rangle|\phi\rangle \\ &\quad = \langle F|I\rangle|\phi\rangle - \frac{igt}{\hbar}\langle F|\hat{A}|I\rangle\hat{P}|\phi\rangle - \cdots.\end{aligned} \tag{6}$$

This leaves the state of pointer after the interaction and post-selection. In the limit of an ideal weak measurement, $gt \to 0$, $|\langle F|I\rangle|^2 = \mathrm{Prob}_{\mathrm{success}}$ is the probability the post-selection succeeds [7]. If we renormalize the state and then truncate the amplitude of each term to lowest order in gt we get

$$|\phi_{fi}\rangle = |\phi\rangle - \frac{igt}{\hbar}\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\hat{P}|\phi\rangle - \cdots, \tag{7}$$

which is just equivalent to dividing by $\langle F|I\rangle = \sqrt{\mathrm{Prob}_{\mathrm{success}}}$. The subscript fi, corresponding to final state $|F\rangle$ and initial state $|I\rangle$, labels the final pointer state, with which we can now calculate the expectation value of $\hat{X}$ of the pointer. The terms which contain an expectation value of an odd number of operators go to zero since the pointer is initially an even function about zero. To first order in gt, the remaining terms give us

$$\begin{aligned}\langle\hat{X}\rangle_{fi} &= \langle\phi_{fi}|\hat{X}|\phi_{fi}\rangle\\ &= \frac{-igt}{\hbar}\,\mathrm{Re}\left(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\right)\langle\phi_{fi}|(\hat{X}\hat{P}-\hat{P}\hat{X})|\phi_{fi}\rangle\\ &\quad+\frac{gt}{\hbar}\,\mathrm{Im}\left(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\right)\langle\phi_{fi}|(\hat{X}\hat{P}+\hat{P}\hat{X})|\phi_{fi}\rangle\\ &= gt\,\mathrm{Re}\left(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\right). \qquad (8)\end{aligned}$$

Here, $\langle\rangle_{fi}$ is used to signify the expectation value of a pointer observable only in the subensemble of measured systems that start in state $|I\rangle$ and are later post-selected in the state $|F\rangle$. Similarly, the momentum expectation value is given by

$$\begin{aligned}\langle\hat{P}\rangle_{fi} &= \langle\phi_{fi}|\hat{P}|\phi_{fi}\rangle\\ &= \frac{-igt}{\hbar}\,\mathrm{Re}\left(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\right)\langle\phi_{fi}|\big(\hat{P}^2-\hat{P}^2\big)|\phi_{fi}\rangle\\ &\quad+\frac{gt}{\hbar}\,\mathrm{Im}\left(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\right)\langle\phi_{fi}|\big(\hat{P}^2+\hat{P}^2\big)|\phi_{fi}\rangle\\ &= \frac{\hbar gt}{2\sigma^2}\,\mathrm{Im}\left(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}\right). \qquad (9)\end{aligned}$$

The shifts from zero in both the $\hat{X}$ and $\hat{P}$ expectation values are proportional to the real and imaginary parts, respectively, of the weak value $\langle\hat{A}\rangle_W$ which is defined as

$$\langle\hat{A}\rangle_W \equiv \frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle}. \qquad (10)$$

In fact, AAV showed that for sufficiently weak coupling $\langle x|\phi_{fi}\rangle$, the final pointer state, will be

$$(\sqrt{2\pi}\,\sigma)^{-1/2}\exp\big(-(x-\langle\hat{A}\rangle_W)^2/4\sigma^2\big),$$

unchanged except for a shift by the weak value.

It has been argued that it is the backaction of the measurement on the measured system that leads to a finite $\mathrm{Im}\langle\hat{A}\rangle_W$ and thus a non-zero $\langle\hat{P}\rangle_{fi}$ [7]. In addition, as the measurement becomes weaker $\langle\hat{P}\rangle_{fi}$ becomes more and more difficult to determine; $\langle\hat{P}\rangle_{fi}$ decreases with gt/σ^2 whereas the width $\Delta\hat{P}$ decreases as $1/\sigma$. Some have gone as far as to define the weak value as $\mathrm{Re}(\frac{\langle F|\hat{A}|I\rangle}{\langle F|I\rangle})$ [11]. Nonetheless, we will show that $\langle\hat{P}\rangle_{fi}$ should not be interpreted as an insignificant artifact of the weak measurement procedure and has an integral role in measuring the Nth-order joint weak value.

One can express the full weak value in terms of the two expectation values of the pointer,

$$\begin{aligned}\langle\hat{A}\rangle_W &= \mathrm{Re}\langle\hat{A}\rangle_W + i\,\mathrm{Im}\langle\hat{A}\rangle_W\\ &= \frac{2\sigma}{gt}\left\langle\frac{1}{2\sigma}\hat{X}+i\frac{\sigma}{\hbar}\hat{P}\right\rangle_{fi}. \qquad (11)\end{aligned}$$

In their derivation of weak values, AAV made the natural choice of a Gaussian for the initial pointer state, as do we. This state also happens to be the ground state $|0\rangle$ of a harmonic oscillator with mass m and frequency ω. For illustration, if one reparameterizes the width of the Gaussian in terms of $m\omega$ such that $\sigma=\sqrt{\hbar/2m\omega}$ it becomes apparent that the operator in the expectation value in Eq. (11) is just the familiar lowering operator,

$$\hat{a}=\sqrt{\frac{m\omega}{2\hbar}}\hat{X}+i\sqrt{\frac{1}{2m\omega\hbar}}\hat{P}. \qquad (12)$$

The operator in Eq. (11) will transform the pointer just as the lowering operator does, even though the pointer is not actually in a harmonic potential. This fact will simplify some of the following calculations. Furthermore, now the weak value can be re-expressed as:

$$\langle\hat{A}\rangle_W = \frac{2\sigma}{gt}\langle\hat{a}\rangle_{fi}. \qquad (13)$$

To our knowledge, this is the first time in the literature that this simple but important relationship between the annihilation operator and weak measurement has been described. The reason the annihilation operator is related to the weak value can be understood as follows. When the coupling is sufficiently weak, the expansion in Eq. (5) shows that the largest pointer amplitude is left unchanged in the ground state. The interaction Hamiltonian shifts some of the pointer state into the first excited state by creating a small amplitude, proportional to $gt\hat{A}$, for the $|1\rangle$ state. If we restrict ourselves to the post-selected subensemble, as in Eq. (7), then this small amplitude changes to be proportional to $gt\langle\hat{A}\rangle_W$. The annihilation operator removes the part of the state that is left unchanged by the coupling, leaving only the shifted component. In other words, the annihilation operator isolates only that part of the pointer state that is changed by the interaction.

We now move on to a derivation of Nth-order joint weak values. In this section, we combine the strategy outlined in the introduction for measuring joint observables with the use of the annihilation operator to extract the weak value. As in previous works, to measure the operator $\hat{M} = \prod_{j=1}^{N} \hat{A}_j$ we apply N separate von Neumann interactions coupling each $\hat{A}_j$ to its own pointer, as in Eq. (2) [22,23]. To simplify the expressions to come we set all g_j to be equal and rewrite the momentum operators $\hat{P}_j$ in terms of the respective raising and lowering operators, $\hat{a}_j^\dagger$ and $\hat{a}_j$, for each of the pointers,

$$\mathcal{H} = i\frac{\hbar g}{2\sigma}\sum_{j=1}^{N}\hat{A}_j\left(\hat{a}_j^\dagger - \hat{a}_j\right). \tag{14}$$

Now we require N different pointers, all beginning in an initial state defined by Eq. (4). The total initial pointer state can be described by the ground state of N harmonic oscillators:

$$|\Phi\rangle = \prod_{j=1}^{N}|\phi_j\rangle = |0\rangle^{\otimes N}. \tag{15}$$

Continuing, using the number-state notation to describe the pointer, we calculate the state of the combined system after the interaction Hamiltonian is applied,

$$\begin{aligned}
|\Phi\rangle|I\rangle &\to \exp\left(\frac{-i\mathcal{H}t}{\hbar}\right)|0\rangle^{\otimes N}|I\rangle \\
&= \left(1 - \frac{i\mathcal{H}t}{\hbar} + \cdots\right)|0\rangle^{\otimes N}|I\rangle \\
&= \left(1 + \frac{gt}{2\sigma}\sum_{j=1}^{N}\hat{A}_j\left(\hat{a}_j^\dagger - \hat{a}_j\right) + \cdots\right)|0\rangle^{\otimes N}|I\rangle \\
&= |0\rangle^{\otimes N}|I\rangle + \frac{gt}{2\sigma}\sum_{j=1}^{N}\hat{A}_j|1_j\rangle|I\rangle + \cdots,
\end{aligned} \tag{16}$$

where $|1_j\rangle$ is the state where the jth pointer is in the first-excited state and all the other pointers are in the ground state (e.g., $|0_1 1_2 0_3 \cdots 0_N\rangle$). Here, we have expanded the state in powers of gt. Eq. (16) shows that to first order, the interaction Hamiltonian coupling the measuring device to the system can displace only one of the N pointers at a time. Simultaneous shifts of multiple pointers come from higher-order terms in the propagator. We are particularly interested in the Nth term in the expansion,

$$\frac{1}{N!}\left(\frac{-i\mathcal{H}t}{\hbar}\right)^N = \frac{1}{N!}\left(\frac{gt}{2\sigma}\sum_{j=1}^{N}\hat{A}_j\left(\hat{a}_k^\dagger - \hat{a}_k\right)\right)^N. \tag{17}$$

This term is the lowest-order one in the expansion which can simultaneously transfer all N pointers into the first excited state (e.g., $|1_1 1_2 1_3 \cdots 1_N\rangle$). This state, which we label as $|1\rangle^{\otimes N}$, is created when each term in the above sum supplies one raising operator. The terms in the sum can contribute the N distinct raising operators in any order and so the portion of Eq. (17) that creates the $|1\rangle^{\otimes N}$ state is equal to

$$\frac{1}{N!}\frac{gt}{2\sigma}\wp\left\{\hat{A}_k\hat{a}_k^\dagger\right\}_N, \tag{18}$$

where $\wp\{\hat{L}_k\}_N$ denotes the sum of all $N!$ orderings of the set of N operators $\{\hat{L}_k\}$. Note that these different orderings are only distinct when the operators do not commute. The remaining portions of Eq. (17) create states where at least one pointer is left in the initial state (e.g., $|2_1 0_2 1_3 \cdots 1_N\rangle$). Projecting onto $\langle F|$ completes the post-selection and leaves us with,

$$\begin{aligned}
&\langle F|\exp\left(\frac{-i\mathcal{H}t}{\hbar}\right)|0\rangle^{\otimes N} \\
&\quad = |0\rangle^{\otimes N}\langle F|I\rangle + \frac{gt}{2\sigma}\sum_{j=1}^{N}\langle F|\hat{A}_j|I\rangle|1_j\rangle + \cdots \\
&\qquad + \left(\frac{gt}{2\sigma}\right)^N\frac{1}{N!}\langle F|\wp\{\hat{A}_k\}_N|I\rangle|1\rangle^{\otimes N} + \cdots.
\end{aligned} \tag{19}$$

We renormalize the resulting N-pointer state $|\Phi_{fi}\rangle$ and then truncate the amplitude of each term at the lowest non-zero order in gt,

$$\begin{aligned}
|\Phi_{fi}\rangle &= |0\rangle^{\otimes N} + \frac{gt}{2\sigma}\sum_{j=1}^{N}\frac{\langle F|\hat{A}_j|I\rangle}{\langle F|I\rangle}|1_j\rangle + \cdots \\
&\quad + \left(\frac{gt}{2\sigma}\right)^N\frac{1}{N!}\frac{\langle F|\wp\{\hat{A}_k\}_N|I\rangle|1\rangle^{\otimes N}}{\langle F|I\rangle} + \cdots.
\end{aligned} \tag{20}$$

This is equivalent to dividing by $\langle F|I\rangle$, the renormalization constant in the limit of no coupling. In analogy with Eq. (13), we now wish to take the expectation value of the product of the annihilation operators for all N pointers,

$$\hat{O} \equiv \prod_{j=1}^{N} \hat{a}_j. \tag{21}$$

In Eq. (20), the $|1\rangle^{\otimes N}$ state is the lowest-order term that does not go to zero when acted on by $\hat{O}$; this term becomes,

$$\hat{O}|\Phi_{fi}\rangle = \left(\frac{gt}{2\sigma}\right)^N \frac{1}{N!} \frac{\langle F|\wp\{\hat{A}_j\}_N|I\rangle}{\langle F|I\rangle} |0\rangle^{\otimes N} + O\big((gt)^{N+1}\big). \tag{22}$$

Clearly, to lowest non-zero order the expectation value then becomes,

$$\begin{aligned}\langle\hat{O}\rangle_{fi} &= \langle\Phi_{fi}|\hat{O}|\Phi_{fi}\rangle \\ &= \langle 0|\left(\frac{gt}{2\sigma}\right)^N \frac{1}{N!} \frac{\langle F|\wp\{\hat{A}_j\}_N|I\rangle}{\langle F|I\rangle}|0\rangle \\ &= \left(\frac{gt}{2\sigma}\right)^N \frac{1}{N!} \frac{\langle F|\wp\{\hat{A}_j\}_N|I\rangle}{\langle F|I\rangle}.\end{aligned} \tag{23}$$

The next lowest-order term in the expectation value corresponds to any of the N pointers undergoing an extra pair of transitions (i.e., a pointer is raised to $|2\rangle$ and subsequently lowered back to $|1\rangle$). Consequently it will be reduced in size by a factor of $2(\frac{gt}{2\sigma})^2$ compared to the lowest-order term. Using Eq. (23) the Nth-order joint weak value can now be expressed in the simple formula

$$\frac{1}{N!}\big\langle\wp\{\hat{A}_j\}_N\big\rangle_W = \left\langle \prod_{j=1}^{N} \hat{a}_j \right\rangle_{fi} \left(\frac{2\sigma}{gt}\right)^N. \tag{24}$$

It is often the case that each operator $\hat{A}_j$ acts on a different particle, ensuring that all $\hat{A}_j$ commute. This allows the further simplification of the Nth-order joint weak value to

$$\left\langle \prod_{j=1}^{N} \hat{A}_j \right\rangle_W = \left\langle \prod_{j=1}^{N} \hat{a}_j \right\rangle_{fi} \left(\frac{2\sigma}{gt}\right)^N. \tag{25}$$

For commuting observables, the magnitude of the simultaneous shift in the N pointers that results from concurrent kicks from all N terms in the Hamiltonian in Eq. (2) is proportional to the shift in one pointer created by a single von Neumann Hamiltonian for measuring operator $\hat{M}$. The role of the annihilation operators is to isolate this simultaneous pointer shift from the total uncorrelated shifts of the N pointers and thus duplicate the action of $\mathcal{H} = g\hat{M}\hat{P}$, without the need for multiparticle interactions.

Since Eq. (25) requires the measurement of the annihilation operator, which is not Hermitian, one might think the expression is, in principle, unmeasurable. In fact, if one expands the annihilation operator in terms of $\hat{X}$ and $\hat{P}$ for each pointer then one is simply left with expectation values of products of $\hat{X}$ or $\hat{P}$ for each pointer. One then measures $\hat{X}$ in one ensemble of pointers and $\hat{P}$ in an identically-prepared ensemble.

The expression in Eq. (25) is the central result of this work. As in previous papers, this result shows how one can practically measure a joint weak value even without the multiparticle interactions the AAV method requires [22,23]. However, this expression is much more elegant and it makes it clear that the annihilation operator plays a key role in joint weak measurements. Specifically, with the use of the annihilation operator, the similarity to the strong measurement expectation value in Eq. (3) is apparent. For strong measurement, the equivalent expectation value to the Nth-order joint weak value is

$$\left\langle \prod_{j=1}^{N} \hat{A}_j \right\rangle = \left\langle \prod_{j=1}^{N} \hat{X}_j \right\rangle \left(\frac{1}{gt}\right)^N. \tag{26}$$

The similarity is striking and makes a good case for the use of the annihilation operator in the understanding of weak values.

Let us compare Eq. (25) to the previous results for the Nth-order joint weak value [23]. In the previous paper, it was expressed recursively in terms of two Nth-order correlations between the pointers and to N different joint weak values of order $N-1$. Utilizing this recursive formula, the Nth-order joint weak value can be expressed purely in terms of the expectation value of position and momentum correlations. This expression includes $2^{N+1}-2$ distinct correlations of various orders, although most will be close to the $N/2$ order as the number of distinct expectation values at each order follows the binomial distribution. In comparison, Eq. (25) relates the Nth-order joint weak value to 2^N correlations in the positions and momenta of all N pointers and so requires roughly half the number of expectation values as the final result from the previous paper (but of higher order).

As a specific example of the use of Eq. (25), the weak value of the product of two spin components $S_{1x}S_{2y}$ would be,

$$\langle S_{1x}S_{2y}\rangle_W = \left(\frac{2\sigma}{gt}\right)^2 \langle \hat{a}_1\hat{a}_2\rangle_{fi} = \left(\frac{2\sigma}{gt}\right)^2 \left\langle \left(\frac{1}{2\sigma}\hat{X}_1 + i\frac{\sigma}{\hbar}\hat{P}_1\right) \times \left(\frac{1}{2\sigma}\hat{X}_2 + i\frac{\sigma}{\hbar}\hat{P}_2\right)\right\rangle_{fi}. \tag{27}$$

The real and imaginary parts of the weak value are then

$$\mathrm{Re}\langle S_{1x}S_{2y}\rangle_W = \left(\frac{1}{gt}\right)^2 \left(\langle \hat{X}_1\hat{X}_2\rangle_{fi} - \frac{4\sigma^4}{\hbar^2}\langle \hat{P}_1\hat{P}_2\rangle_{fi}\right), \tag{28}$$

$$\mathrm{Im}\langle S_{1x}S_{2y}\rangle_W = \frac{2\sigma^2}{\hbar}\left(\frac{1}{gt}\right)^2 \left(\langle \hat{X}_1\hat{P}_2\rangle_{fi} + \langle \hat{P}_1\hat{X}_2\rangle_{fi}\right). \tag{29}$$

The importance of the pointer momentum shift is demonstrated in the above example. With our measurement technique even the real part of weak value is related to the pointers' momenta, $\hat{P}_1$ and $\hat{P}_2$. In general, the momentum and position observables for each of the N pointers will appear in the expression for the real part of the Nth-order joint weak value.

Note that like single weak measurements, this method for measuring the Nth-order joint weak value is not limited to the particular interaction or pointer used in our measurement model [24]. For example, one can perform a derivation very similar to the one presented here where a spin, as opposed to position, pointer is used. For a spin pointer, the Hamiltonian would be $\mathcal{H} = -g\hat{A}\hat{S}_y = ig\hat{A}(\hat{S}_z^+ - \hat{S}_z^-)/2$, where $\hat{S}_i^+$ and $\hat{S}_i^-$ are the raising and lowering operators for the $\hat{S}_i$ basis. The initial pointer state would be the lowest eigenstate of $\hat{S}_z$, with eigenvalue $-\hbar s$. In this case, the expression for the Nth-order joint weak value in terms of N spin pointers is

$$\left\langle \prod_{j=1}^N \hat{A}_j\right\rangle_W = \left\langle \prod_{j=1}^N \hat{S}_{jz}^-\right\rangle_{fi} \left(\frac{1}{gt\hbar s}\right)^N, \tag{30}$$

where $\hat{S}_{jz}^-$ is the z-basis lowering operator for the jth pointer and all $\hat{A}_j$ are assumed to commute. An important advantage of using spin is the absence of unequal coefficients in the expression for the lowering operator. This puts the shifts in the pointer observable and its conjugate on equal footing. Using such a pointer means that the physical shift in the conjugate observable does not become smaller as the measurement becomes weaker. Expectation values are also particularly easy to measure for spins (and polarizations), especially spin $1/2$ systems since there are only two basis states which need to be projected onto. For instance, the Nth-order joint weak value would only require 2^{2N} measurements in total if N spin $1/2$ pointers were used.

In the present work, we have greatly simplified a recent extension of weak measurement which makes the experimental investigation of composite, or joint observables possible [22,23]. We have shown that when single and joint weak values are expressed as expectation values of annihilation operators, they take on a surprisingly elegant form very similar to that seen in standard strong measurement. This form is easily generalized to any measurement device in which the initial pointer state is the eigenstate of an appropriate lowering operator. With the extension, the weak measurement of joint observables only requires the same apparatus that one would need to weakly measure each of the component observables separately. Joint observables are central to the detection and utilization of entanglement in multiparticle systems. The weak measurement of these observables should be particularly useful for investigating post-selected systems such as those that have been used produce novel multiparticle entangled states or those that implement quantum logic gates [25,26].

Acknowledgements

This work was supported by ARC Seibersdorf Research GmbH, the Austrian Science Foundation (FWF), project number SFB 015 P06, NSERC, and the European Commission, contract number IST-2001-38864 (RAMBOQ). We would like to thank Aephraim Steinberg and Morgan Mitchell for helpful discussions.

References

[1] Y. Aharonov, D.Z. Albert, L. Vaidman, Phys. Rev. Lett. 60 (1988) 1351.
[2] A. Peres, Phys. Rev. Lett. 62 (1989) 2326.
[3] A.J. Leggett, Phys. Rev. Lett. 62 (1989) 2325.
[4] Y. Aharonov, L. Vaidman, Phys. Rev. Lett. 62 (1989) 2327.
[5] N. Brunner, A. Acín, D. Collins, N. Gisin, V. Scarani, Phys. Rev. Lett. 91 (2003) 180402;
N. Brunner, V. Scarani, M. Wegmüller, M. Legré, N. Gisin, quant-ph/0407155.
[6] D.R. Solli, C.F. McCormick, R.Y. Chiao, S. Popescu, J.M. Hickmann, Phys. Rev. Lett. 92 (2004) 043601.
[7] A.M. Steinberg, Phys. Rev. Lett. 74 (1995) 2405;
A.M. Steinberg, Phys. Rev. A 52 (1995) 32.
[8] A.M. Steinberg, Can a falling tree make a noise in two forests at the same time?, in: S. Jeffers, G. Hunter, J.-P. Vigier (Eds.), Causality and Locality in Modern Physics, Kluwer, Dordrecht, 1997, p. 431.
[9] L. Hardy, Phys. Rev. Lett. 68 (1992) 2981.
[10] Y. Aharonov, A. Botero, S. Popescu, B. Reznik, J. Tollaksen, Phys. Lett. A 301 (2001) 130.
[11] H.M. Wiseman, Phys. Rev. A 65 (2002) 032111.
[12] G.T. Foster, L.A. Orozco, H.M. Castro-Beltran, H.J. Carmichael, Phys. Rev. Lett. 85 (2000) 3149.
[13] M.O. Scully, B.-G. Englert, H. Walther, Nature 351 (1991) 111.
[14] E.P. Storey, S.M. Tan, M.J. Collett, D.F. Walls, Nature 367 (1994) 626.
[15] H.M. Wiseman, Phys. Lett. A 311 (2003) 285.
[16] L.M. Johansen, Phys. Lett. A 322 (2004) 298.
[17] J. von Neumann, Mathematical Foundations of Quantum Mechanics, Princeton Univ. Press, Princeton, 1955, pp. 442–445.
[18] N.W.M. Ritchie, J.G. Story, R.G. Hulet, Phys. Rev. Lett. 66 (1991) 1107.
[19] K.J. Resch, J.S. Lundeen, A.M. Steinberg, Phys. Lett. A 324 (2004) 125.
[20] A.M. Steinberg, quant-ph/0302003;
J.D. Barrow, P.C.W. Davies, C.L. Harper Jr. (Eds.), Science and Ultimate Reality: Quantum Theory, Cosmology and Complexity, Cambridge Univ. Press, Cambridge, 2004, in press.
[21] K. Mølmer, Phys. Lett. A 292 (2001) 151.
[22] K.J. Resch, A.M. Steinberg, Phys. Rev. Lett. 92 (2004) 130402.
[23] K.J. Resch, J. Opt. B 6 (2004) 482.
[24] L.M. Johansen, quant-ph/0402050.
[25] E. Knill, R. Laflamme, G. Milburn, Nature 409 (2001) 46;
T.B. Pittman, B.C. Jacobs, J.D. Franson, Phys. Rev. A 64 (2001) 062311;
P. Kok, H. Lee, J.P. Dowling, Phys. Rev. A 65 (2002) 052104.
[26] K. Sanaka, T. Jennewein, J.W. Pan, K. Resch, A. Zeilinger, Phys. Rev. Lett. 92 (2004) 017902;
J.W. Pan, D. Bouwmeester, H. Weinfurter, A. Zeilinger, Nature 403 (2000) 515;
S. Gasparoni, J.W. Pan, P. Walther, T. Rudolph, A. Zeilinger, Phys. Rev. Lett. 93 (2004) 020504;
J.L. O'Brien, G.J. Pryde, A.G. White, T.C. Ralph, D. Branning, Nature 426 (2003) 264;
T.B. Pittman, M.J. Fitch, B.C. Jacobs, J.D. Franson, Phys. Rev. A 68 (2003) 032316;
M.W. Mitchell, J.S. Lundeen, A.M. Steinberg, Nature 429 (2004) 161;
P. Walther, J.W. Pan, M. Aspelmeyer, R. Ursin, S. Gasparoni, A. Zeilinger, Nature 429 (2004) 158.